Theoretical Background

Two-Dimensional Photoelasticity

Photoelasticity is a nondestructive, optical technique for experimental stress analysis that is particularly useful for structural components with complex geometric configurations, or subjected to complex loading conditions. Analytical methods of stress analysis are very cumbersome, and often unavailable for such cases, thus amplifying the importance and the need for a suitable experimental approach. Photoelasticity has been used widely, over an extended period of time, for problems in which stress distributions have to be investigated over large sections, or regions, of the structure. It provides quantitative information on highly stressed areas and the associated peak stresses. Equally important is the capability offered by photoelasticity to discern areas of low stress levels, where structural materials are utilized inefficiently.

The method of photoelasticity can be applied in various forms to a wide variety of problems ranging from stress wave propagation to fracture mechanics, to three-dimensional studies. The applications illustrated and practiced in the following experiments are restricted, however, to two-dimensional static problems.

Plane Polariscope

The photoelastic effect is based on the temporary double refraction or temporary birefringence phenomenon, which is exhibited by many transparent polymeric materials when their deformed configurations under stress are observed in polarized light. The birefringence phenomenon is due to the fact that a stressed model of such a material allows two light waves to propagate through it, in other words two different waves are refracted from one incident wave. The directions of the refracted waves are oriented along the principal stress directions, based on the equivalence between the STRESS ELLIPSOID for principal stresses, and the INDEX ELLIPSOID for principal indexes of refraction. The two refracted waves travel at different speeds, one in the FAST direction, and another in the SLOW direction, which creates a RELATIVE RETARDATION (phase difference) that is proportional to the difference between principal stresses, in accordance with the stress-optic law. An instrument called a polarscope can be used to polarize the light in a desired direction, and detect the phase difference introduced by the model between the two refracted waves of polarized light.

For a light beam, when it travels in material:

\[ E = k \sin(\omega t - \Delta_1) \]

When it travels to different material, it will subject to refraction:

\[ \sin \alpha \sin \gamma = \frac{n_2}{n_1} = n_{21}, \text{ where } \]

Where \( n_1 \) and \( n_2 \) are the refraction indexes of material 1 and material 2, respectively.
Where $v$ is the velocity of light in the vacuum, $v_1$ is the velocity of light in material 1, and $v_2$ is the velocity of light in material 2.

Relative angular phase shift caused by birefringence in a WAVE PLATE of thickness "h":

$$\Delta = \frac{2\pi}{\lambda} \delta = \frac{2\pi h}{\lambda} (n_2 - n_1), \text{ where } \lambda \equiv \text{wavelength}, \quad \lambda = \frac{2\pi}{\xi}, \xi \equiv \text{wave number}$$

An applied stress can result in a change in the refractive index, $n$, of a transparent substance. If a general system of stresses is applied in a plane, the optical birefringence, $n_2 - n_1$, will be proportional to the difference, $\sigma_1 - \sigma_2$, between the two principal stresses in the plane. We can define the stress-optical coefficient $C$, such that

$$n_2 - n_1 = C(\sigma_1 - \sigma_2)$$

For a sample of uniform thickness, regions in which $\sigma_1 - \sigma_2$ is constant show the same interference color when viewed between crossed polars.

$$\Delta = \frac{2\pi h c}{\lambda} (\sigma_1 - \sigma_2), \Rightarrow \sigma_1 - \sigma_2 = \frac{Nf_\sigma}{h}$$

where:

$$N = \frac{\Delta}{2\pi}$$

is the fringe order;

$c$ is the relative stress-optic coefficient, expressed in units of "length$^2$/force", or "brewsters";

$f_\sigma$ is material fringe constant, $f_\sigma = \frac{\lambda}{c}$, which is expressed in units of "force/length".

The basic components of a plane polariscope are a light source and two flat plates that form the Linear (Plane) Polarizer/Analyzer combination, consisting of the POLARIZER, placed in front (before) of the model, and the ANALYZER, placed behind (after) the loaded structural model. Each linear polarizer/analyzer unit is made, usually, of a thin Polaroid H film of polyvinyl alcohol, which is heated, stretched, and immediately bonded on a supporting sheet of cellulose acetate butyrate.
Since the intensity of light is proportional to the square of the wave amplitude, the light that emerges from the analyzer of a plane polariscope can be described by the equation:

\[ I = K \sin^2(2\alpha) \times \sin^2\left(\frac{\Delta}{2}\right) \]  

(2)

Where \( \alpha \) is the principle stress direction of \( \sigma_1 \), \( \Delta \) is the phase difference and expressed by,

\[ \Delta = \frac{2\pi h}{\lambda} (n_2 - n_1) = \frac{2\pi hc}{\lambda} (\sigma_1 - \sigma_2) \]

Equation (2) indicates that the intensity will become zero \( (I = 0, \text{black fringes occur}) \) in the following two cases:

a) When \( \sin^2(2\alpha) = 0 \) (which is related to the principle stress direction), **Isoclinic fringe patterns** will occur.

b) When \( \sin^2(\Delta / 2) = 0 \) (which is related to the principle stress difference), **Isochromatic fringe patterns** will occur.

Two families of optical fringes are observed through the plane polariscope as a result of the birefringence phenomenon: **Isoclinics and Isochromatics**.

Black fringes on the ruler [1]
Isoclinics

In order to determine the directions of the principal stress it is necessary to use isoclinic lines as these dark fringes occur whenever the direction of either principal stress aligns parallel to the analyzer or polariser direction. The "isoclinics" are black fringes that describe the loci of constant principal directions, i.e. the lines joining all the points in the model where the orientation of principal stresses is the same. The specific orientation of principal directions corresponding to a certain isoclinic is determined by the specific orientation of the polarizer/analyzer combination, since "$\alpha$" is the angle between the axis of the polarizer and the principal $\sigma_1$ direction. Thus by rotating, in increments, the polarizer/analyzer pair of the polariscope in order to reach the conditions of $2\alpha = n\pi$, $n=0, 1, 2,\ldots$, a whole family of isoclinics may be obtained.

The principal direction corresponding to a certain isoclinic angle is related by Eq.(1) below to the stress components that define the state of stress at any point along that isoclinic:

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}}$$

Isochromatics

The "isochromatics" are lines of constant color which are obtained when a source of white light is used in the polariscope, and they are related to the level of loading. When a monochromatic source is used in the setup, only black fringes are observed, which are labeled by the fringe order, $N=0,1,2,\ldots$, and they are caused by the extinction of the light emerging from the analyzer, as a result of a RELATIVE RETARDATION that meets the condition: $\Delta/2 = n\pi$, $n=0,1,2,\ldots$. The fringe order, "N", of any such isochromatic fringe is related to the stress level at any particular point along that fringe by the stress-optic law:

$$\sigma_1 - \sigma_2 = f_\sigma \frac{N}{h}$$

where $\sigma_1$ and $\sigma_2$ are the principal stresses at that point, $f_\sigma$ is the material fringe constant, $N$ is the fringe order and $h$ is the thickness of the plastic model. Contours of constant principal stress difference are therefore observed as $\sigma_1-\sigma_2$ isochromatic lines. It is obvious from Eq.(4) that a large number of fringes (large fringe orders) indicate regions of high stress in the model. In general, the principal-stress difference and the principal-stress directions vary from point to point in a photoelastic model. As a result the isoclinic fringe pattern and the isochromatic fringe pattern are SUPERIMPOSED when the model is viewed through a PLANE polariscope.

Circular Polariscope

In many cases, simultaneous occurrence of both isoclinics and isochromatics sets of fringes can obscure the interpretation of the fringe pattern. A refinement of the plane polariscope is obtained by inserting quarter wave ($\lambda/4$) plates before and after the model, oriented at an inclination of
45° with respect to the axis of the polarizer. This arrangement, which is illustrated in Fig.1 below, is called a **circular polariscope**, where the isoclinics are removed, leaving only the isochromatics for the analysis of stress levels.

![Fig.1 – Common Schematic of Circular Polariscope](image)

**Fig.1 – Common Schematic of Circular Polariscope**

- S - Light Source
- P - Polarizer
- F - Filter
- \( \lambda/4 \) - Quarter-Wave Plate
- L - Field Lens
- A - Analyzer
- M - Test Specimen
- W - Wall or camera

There are two arrangements for the orientation between the polarizer and the analyzer in order to generate two modes of birefringence: “dark field” and “bright field”.

**Dark field:** If the polarizer is perpendicular to the analyzer (noted as “\( P \perp A \)”), the "Quarter-Wave Plate", which is a particular type of "Wave Plate", is designed such that the birefringence effect generates (in the ABSENCE of any stress) an angular retardation of \( \Delta = \pi/2 \) (Wave Plates that produce angular retardations of \( \Delta = \pi \), or \( 2\pi \), are known as "Half, and Full-Wave" Plates, respectively).

![polarizer and analyzer](image)

The light emerging from the analyzer of a Circular Polariscope in a **Dark Field** arrangement, is given by the following equation:

\[
I = K \sin^2 \left( \frac{\Delta}{2} \right)
\]
\[ \frac{\Delta}{2} = n\pi, \quad \text{for} \quad n = 0, 1, 2, 3, \ldots \quad \text{or} \quad N = \frac{\Delta}{2\pi} = 0, 1, 2, \ldots \] (5)

which is clearly a function ONLY of the principal-stress DIFFERENCE, since the angle "\( \alpha \)" does not appear in the amplitude of the wave. The last equation indicates that the extinction that creates the dark fringes occurs, as in the case of a plane polariscope, when \( \Delta/2 = n\pi, \ n = 0, 1, 2, \ldots \).

**Bright filed:** If the polarizer is parallel to the analyzer (noted as "P//A") in a CIRCULAR Polariscope, similar to that depicted in Fig.1, a bright field will be generated. And then the black fringes observed in the camera "W" correspond to different fringe orders, \( N = 0, 0.5, 1.5, 2.5, \ldots \) since the "fringe order" \( N \), and the integer "\( n \)" no longer coincide. Indeed, the light intensity is proportional to the square of the amplitude of the light-wave, the light emerging from the analyzer in this type of arrangement is given by:

\[ I = K \cos^2\left(\frac{\Delta}{2}\right), \quad \text{which shows that extinction occurs when} \]

\[ \frac{\Delta}{2} = 1 + \frac{2n}{\pi}, \quad \text{for} \quad n = 0, 1, 2, 3, \ldots \quad \text{or} \quad N = \frac{\Delta}{2\pi} = \frac{1}{2} + n \] (6)

Intermediate data can thus be obtained, for smaller increments of stress levels. When BOTH the light- and the dark field arrangements are used in a Circular Polariscope, it is possible to obtain TWO photographs of the resulting fringe patterns, that provide a WHOLE-FIELD representation of the fringe orders, to the nearest \( 1/2 \) order.

**Calibration**

Any photoelastic material must be calibrated before an experiment, in order to obtain the material fringe constant, \( f_{\sigma} \), the simplest approach to the calibration of a new model is the uniaxial tension test of a corresponding specimen, and the following analysis of its results:

\[ \sigma_i = \frac{P}{wh}, \quad \sigma_z = 0 \quad w = \text{width} \]

giving

\[ P = f_{\sigma} wN \] (7)
Therefore, "$f_\sigma$" can be obtained from the slope of the graph of applied load versus fringe order.

**Photoelastic Analysis**

A photoelastic model that is geometrically similar to the actual structural component is made of a proper transparent polymer, and then observed in the polariscope, under the proper loading conditions. The fringes of integer order are sketched or photographed, and labeled. If the accuracy of the measured stress levels is important, then the half order fringe data are also collected. Based on the stress-optic relationship, the difference between the principal stresses (or the maximum shear stress) can be determined at any point in the body.

Additional information is required if the principal stress components have to be determined separately (rather than merely the difference between them). At free boundaries the separated stresses are obtained immediately since one of the principal stresses is zero. A beam in pure bending can be assumed to experience only uniaxial stress, since the bending moment is uniform.

**Stress Concentrations**

Since the method of photoelasticity is not restricted to simple geometric configurations, nor to regular boundary shapes, it is applied widely and effectively to analyzing stress concentrations, and calculating stress concentration factors around holes and other geometric discontinuities. Furthermore, the point of maximum stress does not have to be known a priori, since photoelasticity provides stress distributions over a finite area. The separate components of principal and maximum shear stresses can be obtained directly from the fringe data, without the need for tedious strain/stress transformations.

A geometric discontinuity (hole or notch, for example, Fig.2) in a body causes a localized increase in stress, even though the externally applied loads remain unchanged. The stress concentration factor is defined as the ratio between the maximum local stress around the region of geometric discontinuity, and to the nominal, reference or undisturbed stress in the structure, in regions that are sufficiently remote from the discontinuity. It is calculated through the following equations:

\[
K_i = \frac{\sigma_{\text{max}}}{\sigma_{\text{nom}}} \quad (8)
\]

Specially,

\[
K_{\text{tg}} = \frac{\sigma_{\text{max}}}{\sigma_{\text{norm}_{\text{gross}}}}, \quad (9)
\]

where \[
\sigma_{\text{max}} = \frac{f_\sigma \cdot N}{h} \]

\[
\sigma_{\text{norm}_{\text{gross}}} = \frac{P}{hw} \quad \text{is the gross remote stress}
\]
$K_m = \frac{\sigma_{\text{max}}}{\sigma_{\text{norm,net}}}$, where $\sigma_{\text{norm,net}} = \frac{P}{h(w-2r)}$ is the net remote stress \hfill (10)

$\sigma_{\text{max}}$ is determined from fringe patterns. Hence, $K_m$ can be determined by the photoelastic experiment.

From equation (9) and (10),

$$K_{tp} = K_m \frac{w}{w-2r}$$ \hfill (11)

In the above equations, $P$ is the applied load, $h$ is the specimen thickness, $w$ is the specimen width and $r$ is the radius of the notch. The choice of reference stress is not always obvious or unique, so when a stress concentration factor is stated, it is also necessary to define the state of reference stress.

For a strip with double notched in tension (Fig.2), the theoretical stress concentration $K_m$ is expressed by:

$$K_m = 3.065 - 3.472\left(\frac{2r}{w}\right) + 1.009\left(\frac{2r}{w}\right)^2 + 0.405\left(\frac{2r}{w}\right)^3$$

![Fig.2 - Photoelastic Test Specimens Made of Homolite 100](image-url)
Another useful test configuration is Four-Point Bending Beam system (Fig. 3), since the bending moment is constant between the innermost loads. To show this we consider moment balance in the region between the innermost loads:

\[ M + \frac{P}{2} (x - d) - \frac{P}{2} \cdot x = 0 \]

Hence, the moment acting is \( M = \frac{P d}{2} \)

The beam experiences a pure bending moment. The magnitude of the bending stress, \( \sigma \), at distance \( y \) from the neutral axis is

\[ \sigma(x) = \frac{M y}{I} \]
\[ \sigma(y) = 0 \]
\[ \sigma(xy) = 0 \]

Where \( I \) is the beam moment of inertia.

\[ I = \frac{1}{12} bh^3 \]

In the beam, the longitudinal stress \( \sigma(x) \) is a principal stress, there are no vertical or horizontal shear stresses and no transverse normal stresses. Therefore, if at a point \( P \) a distance \( y \) from the neutral axis there exists a stress which would cause a phase difference of one wave length, then all points on the horizontal line through \( P \) parallel to the longitudinal axis of the beam would cause the same phase difference.