Pressure Vessels and Strain Transformation

Background

Thin-walled pressure vessels are commonly employed in mechanical and aerospace engineering applications. Some examples are fuel tanks, air tanks and numerous applications where high pressure storage of gas or liquid is required. A high-altitude fuselage can also be modeled as a thin-walled pressure vessel. The fuselage is pressurized to maintain a normal ambient pressure inside for the crew and passengers. Loading due to pressurization may be large, especially near areas of cutouts such as windows. Below we will consider a thin-walled cylinder, which is closed at its ends under internal pressure.

Pressure Vessel Theory - Equilibrium Equations

The following assumptions should be made with regard to a thin-walled pressure vessel:
1. Because the walls are thin, it is assumed that they do not carry transverse load,
2. Because of symmetry, the forces are assumed to be uniformly distributed.

Consider a pressurized, thin-walled cylinder that is closed at both ends, and it is subjected to a uniform internal pressure $P$.

Force equilibrium condition in the longitudinal direction:

$$\sum F_z = 0 \quad \Rightarrow \quad -F_z + p\pi \frac{D^2}{4} = 0 \quad \Rightarrow \quad F_z = P\pi \frac{D^2}{4}$$

Force equilibrium condition in the hoop (or circumferential) direction:

$$\sum F_h = 0 \quad \Rightarrow \quad 2F_h - p \cdot D \cdot \Delta L = 0 \quad \Rightarrow \quad F_h = PD\frac{\Delta L}{2}$$

Therefore, the stresses are,
The stresses $\sigma_z$ and $\sigma_h$ are called the **longitudinal** and **hoop** (or circumferential) stresses, respectively. Because of symmetry, they are the **principal stresses**. It should also be noted that internal pressure causes a combined loading stress state.

Under elastic deformation, the strain can be obtained using Hooke’s law:

$$\varepsilon_z = \frac{1}{E} (\sigma_z - \nu \sigma_h)$$

$$\varepsilon_h = \frac{1}{E} (\sigma_h - \nu \sigma_z)$$

All shear strains are equal to zero.

### Strain Measurement of Thin-walled Cylinder

Two-dimensional strain transformation equations allow the calculation of the strain components in any arbitrary direction. Using strain transformation equations, strain measured with respect to one coordinate axis can be used to obtain strain components with respect to another coordinate axis.

![Strain Diagram](image)

The normal strain at any arbitrary angle of $\theta$ can be obtained:

$$\varepsilon_\theta = \varepsilon_z \cos^2 \theta + \varepsilon_h \sin^2 \theta + \gamma_{xy} \sin \theta \cos \theta$$

For thin-walled cylinder, from strength of materials, we know there are only normal strains on the external free surface. $\gamma_{xy} = 0$. Hence,
\[ \varepsilon_\theta = \varepsilon_x \cos^2 \theta + \varepsilon_y \sin^2 \theta \]

This is the **theoretically expected normal strain component** at any given angle \( \theta \).